Prime numbers and the (double) primorial sieve.

1. The quest.

This research into the distribution of prime numbers started by placing a prime number on the short leg of a Pythagorean triangle. The properties of the Pythagorean triples imply that prime numbers $> p_4$ (with $p_4 = 7$) are not divisible by $p_i \in \{2, 3, 5, 7\}$. These divisors have a combined repeating pattern based on the primorial $p_4\#$, the product of the first four prime numbers. This pattern defines the fourth (double) primorial sieve.
The $P_4\#$–sieve is the first sieve with all the properties of the infinite set of primorial sieves. It also gives an explanation for the higher occurrence of the (9, 1) last digit gap among prime numbers.

The (double) primorial sieve has two main functions.
The primorial sieve is a method to generate a consecutive list of prime numbers, since the base of the n-th primorial sieve contains all prime numbers $< p_n\#$.
The double primorial sieve offers preliminary filtering of natural numbers when they are sequential stacked above the base of the sieve. Possible prime numbers are only found in the columns supported by the $\varphi(p_n\#)$ struts.
Each strut is relative prime to $p_n\#$ and is like a support beam under a column.

Fig. 1: Examples of imaginary struts to support a specific column.

2. The principal of the (double) primorial sieve.

The width of the primorial sieve $P_n\#$−sieve is determined by the primorial $p_n\#$, the product of the first n prime numbers. All natural numbers $g > p_n\#$ form a matrix of infinite height when stacked sequential on top of the base of the sieve (Fig. 2abc). The $\varphi(p_n\#)$ struts support the columns that contain possible prime numbers.
Natural numbers $> p_n\#$ in the unsupported columns are filtered and definitively composite numbers.
For the struts S_j applies that $gcd\,(S_j,\ p_n\#) = 1$ with $1 \le j \le \varphi(p_n\#)$ and $\varphi(m)$ Euler's totient function.
The list of all prime numbers $< p_n\#$ consist of the prime numbers $\le p_n$ and the non-composite struts.

Each primorial sieve is constructed from the previous sieve and therefore from all previous sieves.
In order to generate the $P_1\#$−sieve the $P_0\#$−sieve is defined, based on the "prime number" $p_0 = 1$.
The $P_0\#$−sieve (Fig. 2a) has a width of $p_0\# = 1$ and $\varphi(p_0\#) = 1$ strut. All natural numbers $g > p_0\#$ are situated above the one strut S_1 and are possible prime; there is no filtering.

The $P_1\#$−sieve (Fig. 2b) has a width of $p_1\# = p_1 = 2$ and filters out the even numbers via $g \equiv 0 \pmod{p_1}$.
Natural numbers $g > p_1\#$ above the $\varphi(p_1\#) = 1$ strut S_1 are possible primes.

The $P_2\#$−sieve (Fig. 2c) has a width of $p_2\# = p_2 \bullet p_1\# = 6$ and filters $g \equiv 0 \pmod{p_2}$. Natural numbers $g > p_2\#$ above the $\varphi(p_2\#) = 2$ struts $S(p_2\#)_i \in \{S_1\,,\,S_2\}$ comply with $gcd\,(g,\ p_2) = 1$ as well as $gcd\,(g,\ p_2\#) = 1$ due to the construction via all previous sieves. Possible prime numbers $g > p_2\#$ are of the form $6a \pm 1$ with $a \in \mathbf{N}^{+}$.

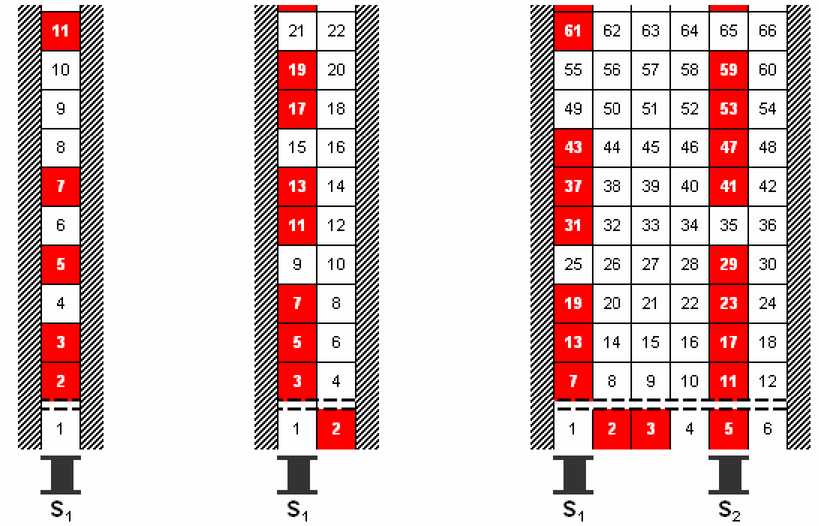

Fig. 2abc: The double primorial sieves: $P_0\#$−sieve, $P_1\#$−sieve and $P_2\#$−sieve.

The third (double) primorial sieve.

From the $P_3\#$−sieve onwards there is an unambiguous algorithm to build the struts from the previous sieve.
The $S(p_2\#)_i$ struts of the $P_2\#$−sieve are repeated p_3 times, so that $S(p_3\#)_k = S(p_2\#)_i + m \bullet p_2\#$ with $0 \le m < p_3$.
Struts with $S(p_3\#)_k \equiv 0 \pmod{p_3}$ are removed (Fig. 3) and thus (multiples of) $S(p_3\#)_2$.

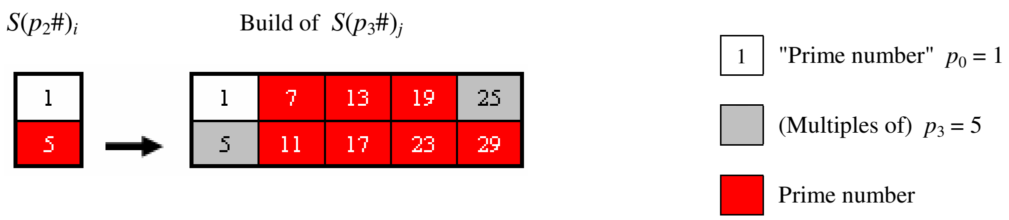

Fig. 3: $P_3\#$−sieve: generating the struts.

The $P_3\#$−sieve has a width of $p_3\# = p_3 \bullet p_2\# = 30$ and $\varphi(p_3\#) = 8$ struts. The $P_3\#$−sieve provides the list of prime numbers $< p_3\#$ consisting of the prime numbers $p_i \in \{2, 3, 5\}$ and the struts S_j that satisfy $gcd\,(S_j,\ p_3\#) = 1$ with $1 < j \leq \varphi(p_3\#)$. Note that with the third primorial sieve all struts > 1 are prime numbers. Potential prime numbers $> p_3\#$ are situated above the struts and meet both $gcd\,(g,\ p_3) = 1$ and $gcd\,(g,\ p_3\#) = 1$ (Fig. 4a).

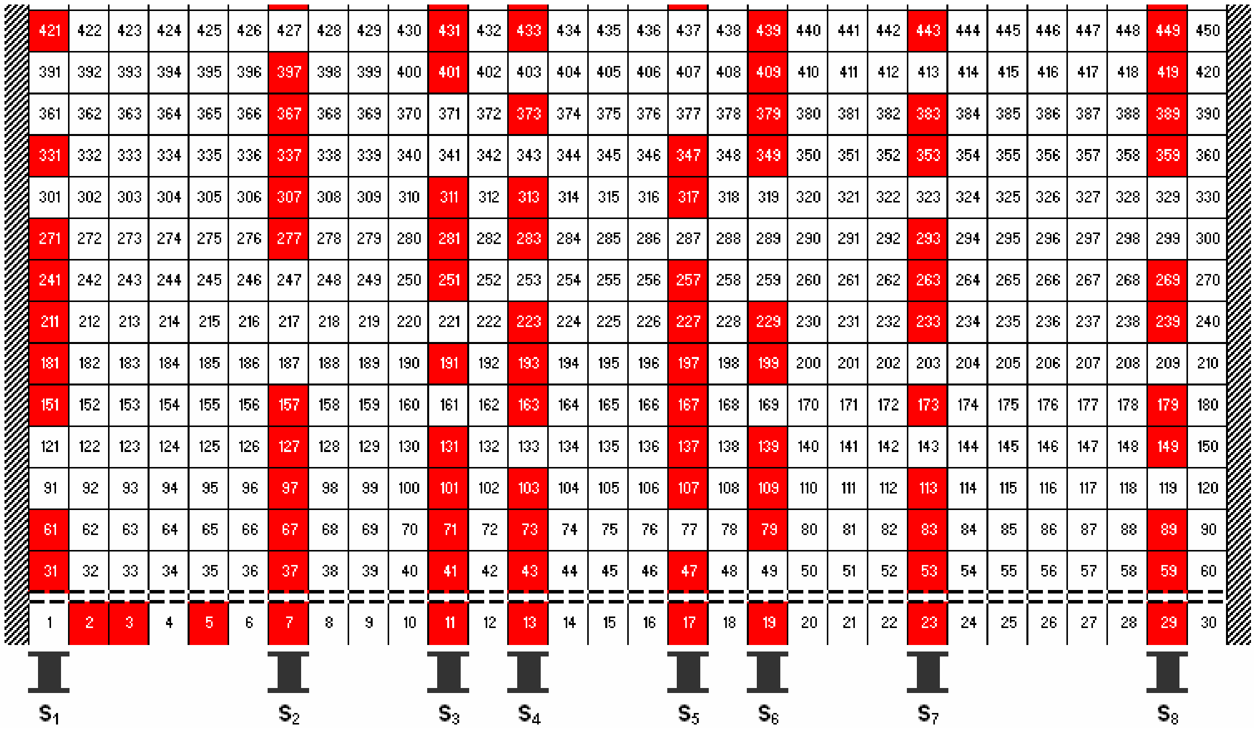

Fig. 4a: The third double primorial sieve.

The $P_3\#$−sieve has many similarities with the Wheel Factorization method of Paul Pritchard. Fig. 4b shows a wheel with the inner circle formed by the first 30 natural numbers, and thus with a $p_3\# = 30$ base. The spokes of the wheel that contain possible prime numbers have the same functionality as the columns above the struts of the primorial sieve.
The graphical representation of the wheel is in this case more concrete. Clearly visible is the symmetry of the spokes in $p_3\# / 2$.

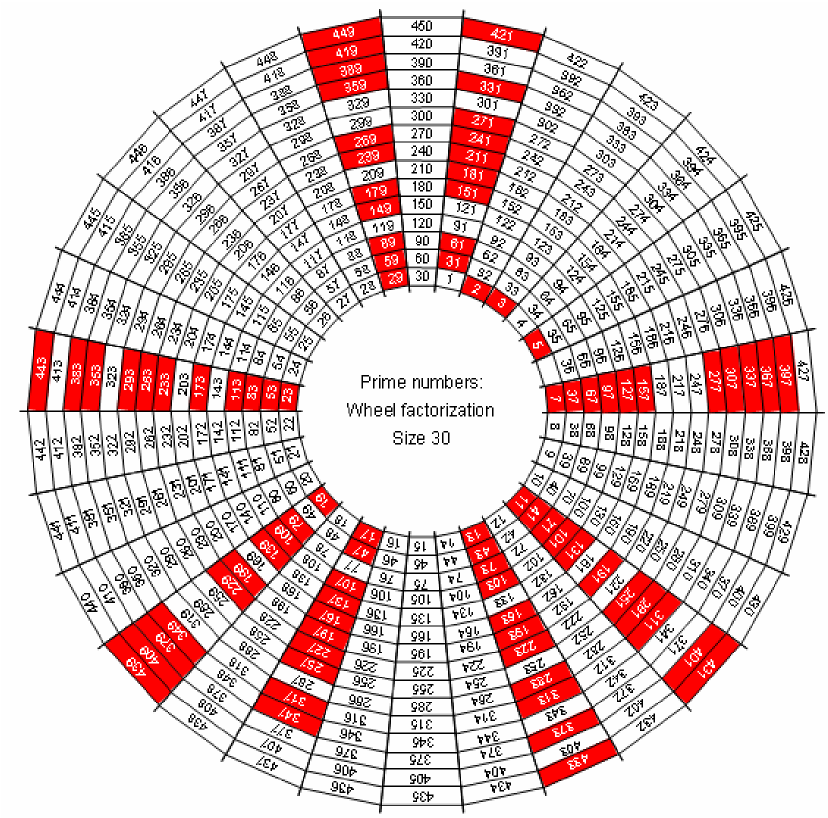

Fig. 4b: Wheel factorization with size 30.

The fourth double primorial sieve.

The struts of the $P_4\#$–sieve are built via the $P_3\#$–sieve by repeating the pattern of the $\varphi(p_3\#) = 8$ struts p_4 times, so that $S(p_4\#)_k = S(p_3\#)_i + m \bullet p_3\#$ with $0 \le m < p_4$. The struts to be removed with $S(p_4\#)_k \equiv 0 \pmod{p_4}$, consist of $p_4 \bullet S(p_3\#)_i$ with $1 \le i \le \varphi(p_3\#)$ (Fig. 5). The $P_4\#$–sieve has $j = \varphi(p_4\#) = (p_4 - 1) \bullet \varphi(p_3\#) = 48$ struts.
The $P_4\#$–sieve is the first sieve with all the characteristics of the double primorial sieve, since the struts are no longer exclusively prime numbers, for example $S(p_4\#)_{39} = 169 = 13 \bullet 13$.

To generate the list $> p_4$ with all prime numbers $< p_4\#$ from the struts of the $P_4\#$–sieve the composite struts $S(p_4\#)_j$ with $j \in \{28, 33, 39, 43, 48\}$ are marked negative (Fig. 5).
These composite struts are found via $S(p_4\#)_j \bullet S(p_4\#)_i < p_4\#$ with $i,\ j > 1$ and $S(p_4\#)_j \le S(p_4\#)_i$.
Thus: **11** • 11 = 121, **11** • 13 = 143, **11** • 17 = 187, **11** • 19 = 209 and **13** • 13 = 169.
The prime numbers $\le p_4$ plus the non-composite struts $> p_4$ supply the list of the 46 prime numbers $< p_4\#$.

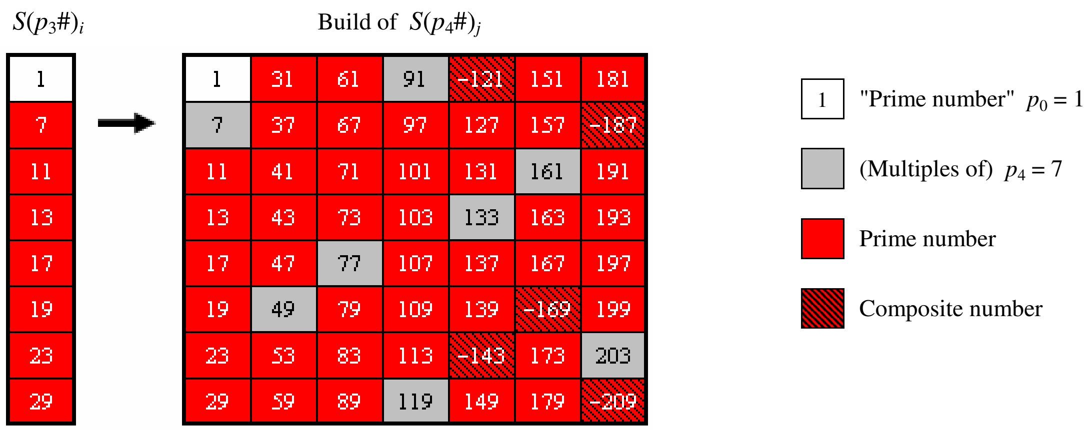

Fig. 5: $P_4\#$–sieve: building the struts

Fig. 6 shows the equal distribution of the $\pi\,(10^9) = 50{,}847{,}534$ prime numbers above the struts of the $P_4\#$–sieve, with a deviation relative to $\pi\,(10^9)\,/\,\varphi(p_4\#)$ of less than 0,05%. Among the struts of the $P_4\#$–sieve the influence is still visible of the repeated pattern of the 8 struts {1, 7, 11, 13, 17, 19, 23, 29} from the $P_3\#$–sieve.
The distance d between S_1 and S_2 of the $P_4\#$–sieve equals to $d = S(p_4\#)_2 - S(p_4\#)_1 = p_5 - p_0 = 11 - 1 = 10$.
This is the biggest gap between the struts (Fig. 5 and Fig. 6). Due to the symmetry in $(p_4\# / 2)$ the distance d is also found between the second to last and the last strut of the sieve.

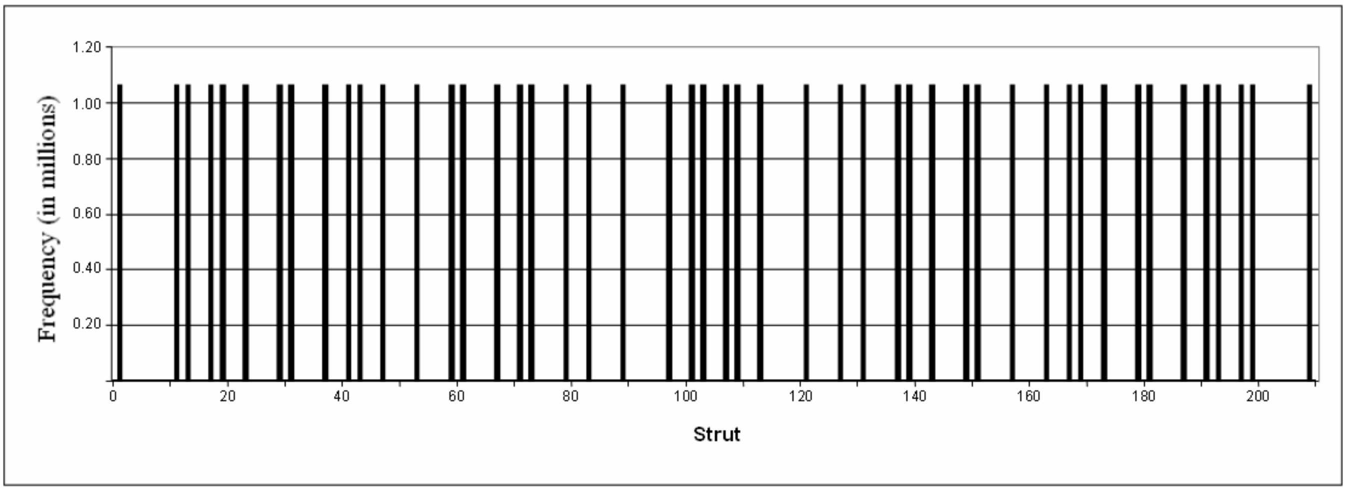

Fig. 6: $P_4\#$–sieve: distribution of prime numbers < 10^9.

3. The double primorial sieve in practical applications.

3.1.1. Example of how to implement the $P_4\#$−sieve.

Each double primorial sieve with the base of $p_n\#$ can be used to determine whether a natural number g is a prime number. The following distinction is made:

- A number $g \le p_n$ is a prime number when $g \in \{p_1, p_2, \dots , p_n\}$
- A number $p_n < g \le p_n\#$ is a prime number when $g \in \{ S(p_n\#)_j \mid gcd\,(g, p_k) = 1$ for all $p_n < p_k \le \sqrt{g}\}$.
  The values of p_k correspond with the struts $p_n < S(p_n\#)_j \le \sqrt{g}$ that are not composite.
- A number $p_n\# < g \le (p_n\#)^2$ can be a prime number when $g \pmod{p_n\#} \in \{ S(p_n\#)_j \}$, and thus
  $g \in \{ S(p_n\#)_j + m \bullet p_n\# \mid 1 \le j \le \varphi(p_n\#) \wedge m \in \mathbf{N}^+ \}$
  The number g is a prime number as further applies that $gcd\,(g, p_k) = 1$ for all $p_n < p_k \le \sqrt{g}$.
  The values of p_k correspond with the struts $p_n < S(p_n\#)_j \le \sqrt{g}$ that are non-composite.
- A number $g > (p_n\#)^2$ can be a prime number when $g \pmod{p_n\#} \in \{ S(p_n\#)_j \}$.
  Checking for possible prime of $g > (p_n\#)^2$ via $p_k \mid g$ for all $p_n < p_k \le \sqrt{g}$ can be approximated by $d \mid g$ with
  $d \in \{ S(p_n\#)_j + m \bullet p_n\# \mid 1 \le j \le \varphi(p_n\#) \wedge m \in \mathbf{N}_0 \}$
  For $d < p_n\#$ division by the struts $> p_n$ that are non-composite suffices.

To determine whether $g \pmod{p_n\#} \in \{ S(p_n\#)_j \}$ the property can be used that with bigger $P_n\#$−sieves the $\varphi(p_n\#)$ struts $S(p_n\#)_j$ are almost proportional split over the $p_n\#$−base.
For example: Fig. 7a is based on the $P_6\#$−sieve with $p_6\# = 30030$ and $(p_6\#)^2 \approx 9 \bullet 10^8$.
The natural number $g_1 = 20893$ is near the strut $g_1 \bullet \varphi(p_6\#) / p_6\# = 4007$ of the $P_6\#$−sieve and ultimately coincides with strut 4009, that is marked as composite. The prime factors are not found.

The natural numbers $g_2 = g_1 + 10^3 \bullet p_6\# < (p_6\#)^2$ and $g_3 = g_1 + 10^5 \bullet p_6\# > (p_6\#)^2$ with $g_2 \equiv g_3 \equiv g_1 \pmod{p_6\#}$ are possible prime, since both numbers coincides with the strut 4009 (even though the strut is composite).
The list of prime numbers up to $\sqrt{g_2}$ from the base of the sieve suffices to find a prime divisor of g_2.
Examining g_3 starts with all prime numbers from the base of the sieve, e.g. the struts $p_6 < S(p_6\#)_j \le p_6\#$ that are non-composite. Prime divisors $> p_6\#$ are approximated by $S(p_6\#)_j + m \bullet p_6\#$ with $m \ge 1$.

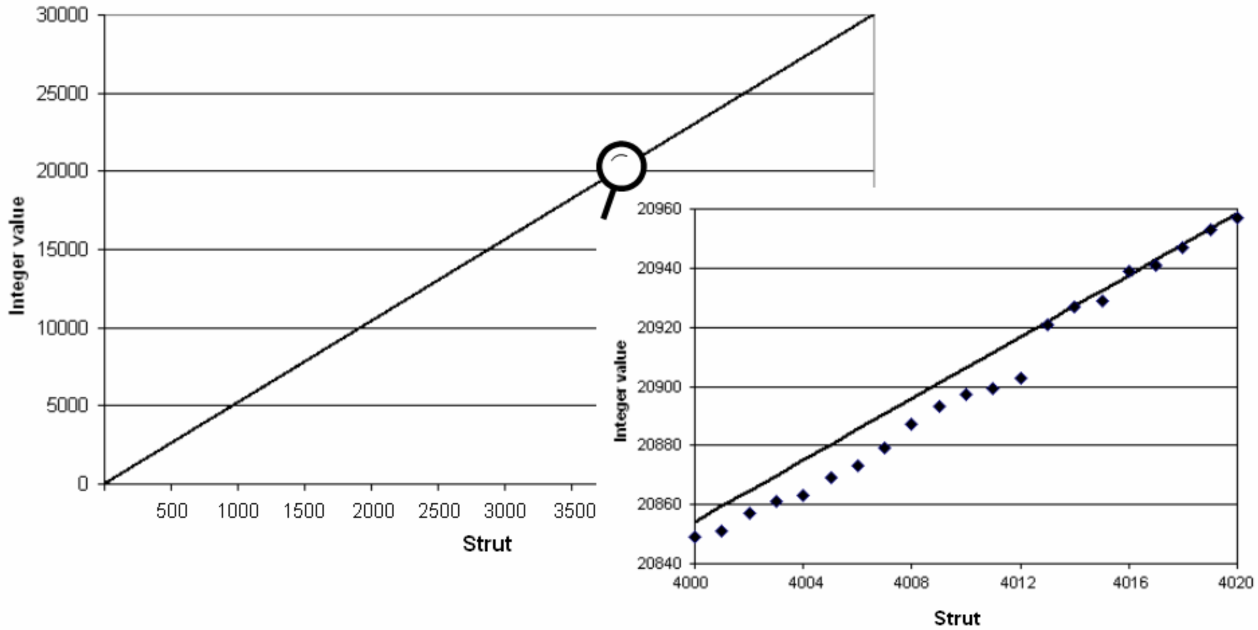

Fig. 7a: $P_6\#$−sieve: the distribution of the struts over the base.

3.1.2. Example of how to implement the $P_4\#$−sieve.

Fig 7b below shows the $P_4\#$−sieve with a base of $p_4\# = p_1 \bullet p_2 \bullet p_3 \bullet p_4 = 210$ and $(p_4\#)^2 = 44{,}100$. Possible prime numbers $> p_4$ (with $p_4 = 7$) are only found above the $\varphi(p_4\#) = 48$ struts of the $P_4\#$−sieve.

89251	89252	89253	89254	89255	89256	89257	…	**89417**	89418	89419	89420	89421	89422	89423	89424	89425	89426	89427	89428	89429	89430	**89431**	…
89041	89042	89043	89044	89045	89046	89047	…	89207	89208	**89209**	89210	89211	89212	**89213**	89214	89215	89216	89217	89218	89219	89220	89221	…
88831	88832	88833	88834	88835	88836	88837	…	**88997**	88998	88999	89000	89001	89002	**89003**	89004	89005	89006	89007	89008	**89009**	89010	89011	…
88621	88622	88623	88624	88625	88626	88627	…	88787	88788	**88789**	88790	88791	88792	**88793**	88794	88795	88796	88797	88798	**88799**	88800	**88801**	…
88411	88412	88413	88414	88415	88416	88417	…	88577	88578	88579	88580	88581	88582	88583	88584	88585	88586	88587	88588	**88589**	88590	**88591**	…
…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…
27931	27932	27933	27934	27935	27936	27937	…	**28097**	28098	**28099**	28100	28101	28102	28103	28104	28105	28106	28107	28108	**28109**	28110	**28111**	…
27721	27722	27723	27724	27725	27726	27727	…	27887	27888	27889	27890	27891	27892	**27893**	27894	27895	27896	27897	27898	27899	27900	**27901**	…
27511	27512	27513	27514	27515	27516	27517	…	27677	27678	27679	27680	27681	27682	27683	27684	27685	27686	27687	27688	**27689**	27690	**27691**	…
27301	27302	27303	27304	27305	27306	27307	…	27467	27468	27469	27470	27471	27472	27473	27474	27475	27476	27477	27478	**27479**	27480	**27481**	…
27091	27092	27093	27094	27095	27096	27097	…	27257	27258	**27259**	27260	27261	27262	27263	27264	27265	27266	27267	27268	27269	27270	**27271**	…
…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…
1471	1472	1473	1474	1475	1476	1477	…	**1637**	1638	1639	1640	1641	1642	1643	1644	1645	1646	1647	1648	1649	1650	1651	…
1261	1262	1263	1264	1265	1266	1267	…	**1427**	1428	**1429**	1430	1431	1432	**1433**	1434	1435	1436	1437	1438	**1439**	1440	1441	…
1051	1052	1053	1054	1055	1056	1057	…	**1217**	1218	1219	1220	1221	1222	**1223**	1224	1225	1226	1227	1228	**1229**	1230	**1231**	…
841	842	843	844	845	846	847	…	1007	1008	**1009**	1010	1011	1012	**1013**	1014	1015	1016	1017	1018	**1019**	1020	**1021**	…
631	632	633	634	635	636	637	…	**797**	798	799	800	801	802	803	804	805	806	807	808	**809**	810	**811**	…
421	422	423	424	425	426	427	…	**587**	588	589	590	591	592	**593**	594	595	596	597	598	**599**	600	**601**	…
211	212	213	214	215	216	217	…	377	378	**379**	380	381	382	**383**	384	385	386	387	388	**389**	390	391	…
1	**2**	**3**	4	**5**	6	**7**	…	**167**	168	169	170	171	172	**173**	174	175	176	177	178	**179**	180	**181**	…
S_1								S_{38}		S_{39}				S_{40}						S_{41}		S_{42}	

Fig. 7b: $P_4\#$−sieve: the distribution of prime numbers above the struts.

The $P_4\#$−sieve can be used to determine whether a natural number g is a prime number.
The following distinctions are made:

- A number g with $g \le p_4$ is a prime number when $g \in \{p_1, p_2, p_3, p_4\}$
- A number g with $p_4 < g \le p_4\#$ is a prime number when $g \in \{\, S(p_4\#)_j \mid gcd\,(g, p_k) = 1$ for all $p_4 < p_k \le \sqrt{g}\}$.
  The values of p_k correspond with the struts $p_4 < S(p_4\#)_j \le \sqrt{g}$ that are not composite.

  $g = 169$ (= $S(p_4\#)_{39}$) → Composite since $d \mid g$ for $d = 13$

- A number g with $p_4\# < g \le (p_4\#)^2$ can be a prime number when $g \pmod{p_4\#} \in \{\, S(p_4\#)_j \,\}$, and thus
  $g \in \{\, S(p_4\#)_j + m \bullet p_4\# \mid 1 \le j \le \varphi(p_4\#) \wedge m \in \mathbf{N}^+ \}$
  The number g is a prime number as further applies that $gcd\,(g, p_k) = 1$ for all $p_4 < p_k \le \sqrt{g}$.
  The values of p_k correspond with the struts $p_4 < S(p_4\#)_j \le \sqrt{g}$ that are non-composite.

  $g = 27{,}469$ → Composite since $d \mid g$ for $d = 13$
  $g = 27{,}679$ → Composite since $d \mid g$ for $d = 89$
  $g = 27{,}889$ → Composite since $d \mid g$ for $d = 167$ ← $d = 169$ will not happen since $13 \mid 169$

- A number g with $g > (p_4\#)^2$ can be a prime number when $g \pmod{p_4\#} \in \{\, S(p_4\#)_j \,\}$.
  Checking for possible prime of $g > (p_4\#)^2$ via $p_k \mid g$ for all $p_4 < p_k \le \sqrt{g}$ can be approximated by $d \mid g$ with $d \in \{\, S(p_4\#)_j + m \bullet p_4\# \mid 1 \le j \le \varphi(p_4\#) \wedge m \in \mathbf{N}_0 \}$
  For $d < p_4\#$ division by the struts $> p_4$ that are non-composite suffices.

  $g = 88{,}579$ → Composite since $d \mid g$ for $d = 283$
  $g = 88{,}999$ → Composite since $d \mid g$ for $d = 61$

3.2.1. The gap between prime numbers.

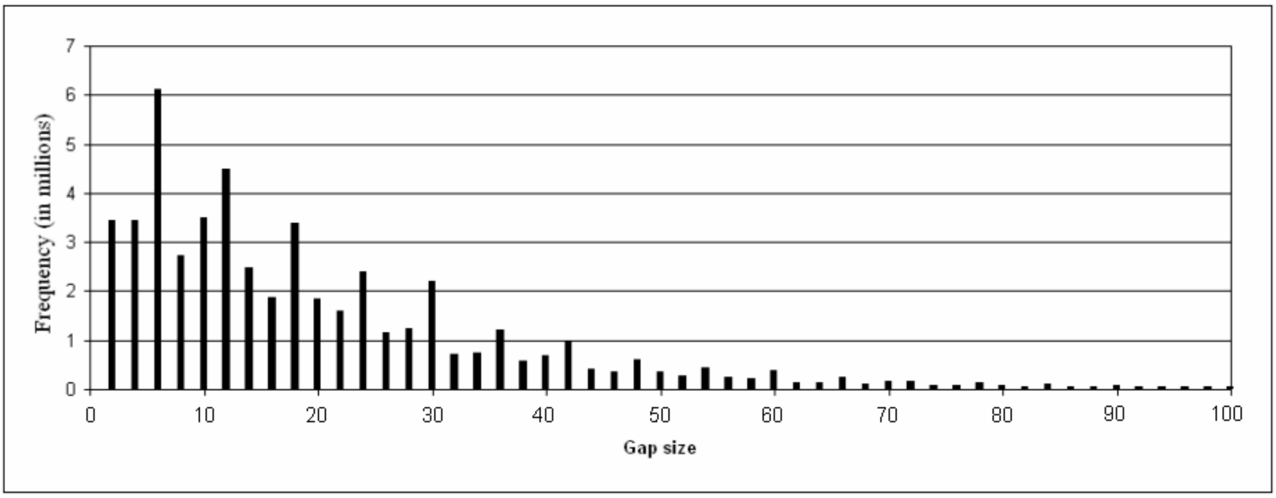

Fig. 8a: Distribution of the prime gaps for all prime numbers up to 10^9.

Fig. 8a shows the distribution of the prime gaps (difference between two consecutive prime numbers) for all prime numbers $< 10^9$. The bar chart gives local maxima at multiples of 6.

The $P_2\#$–sieve divides possible prime numbers into two groups above the struts S_1 and S_2, based on $6a \pm 1$ with $a \in \mathbf{N}^+$ (fig. 2c). Each group is split $p_3 = 5$ ways when building the $P_3\#$–sieve. One of the new columns is (a multiple of) p_3 and contains no prime numbers $> p_3$ (fig. 3 and 4a).
Fig. 8b shows the combined distribution of the prime gaps per separate group, with each group contributing equally. The ratio of prime numbers in $f(a) = 6a + 1$ corresponds with three times the ratio of prime numbers in $f(n) = n$. Clearly visible is the $\varphi(p_2\#) / \varphi(p_3\#)$ increase at (multiples of) $p_3\# = 30$. The higher frequency at gap = 42 is credited to a higher sieve.

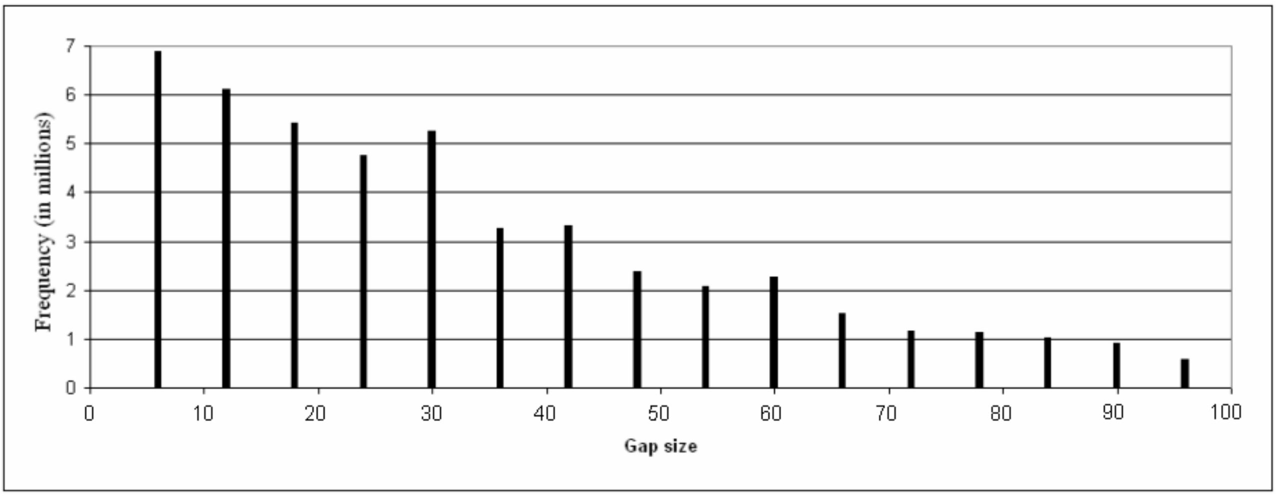

Fig. 8b: Distribution of the prime gaps for all prime numbers up to 10^9, when split into the $6a \pm 1$ groups.

Fig. 8c gives the distribution of prime numbers < 10^9 above the struts of the P_4#−sieve. The columns are set apart based on $6a + 1$ and $6a - 1$ with $a \in \mathbf{N}^+$ (see also Fig. 6). Together the struts show no apparent pattern, simultaneously they are uniquely mirrored in $p_4\# / 2$. Dividing the prime numbers into two groups supports the conjecture that twin primes are not related by a common denominator.
At the same time the P_2#−sieve stipulates that all twin primes $> p_2$ are of the form $6a \pm 1$ (fig. 2c). Based on the P_3#−sieve twin primes $> p_3$ are narrowed down to the form $(30a + 12m) \pm 1$ with $a \in \mathbf{N_0}$ and $-1 \le m \le 1$ (Fig. 4b).

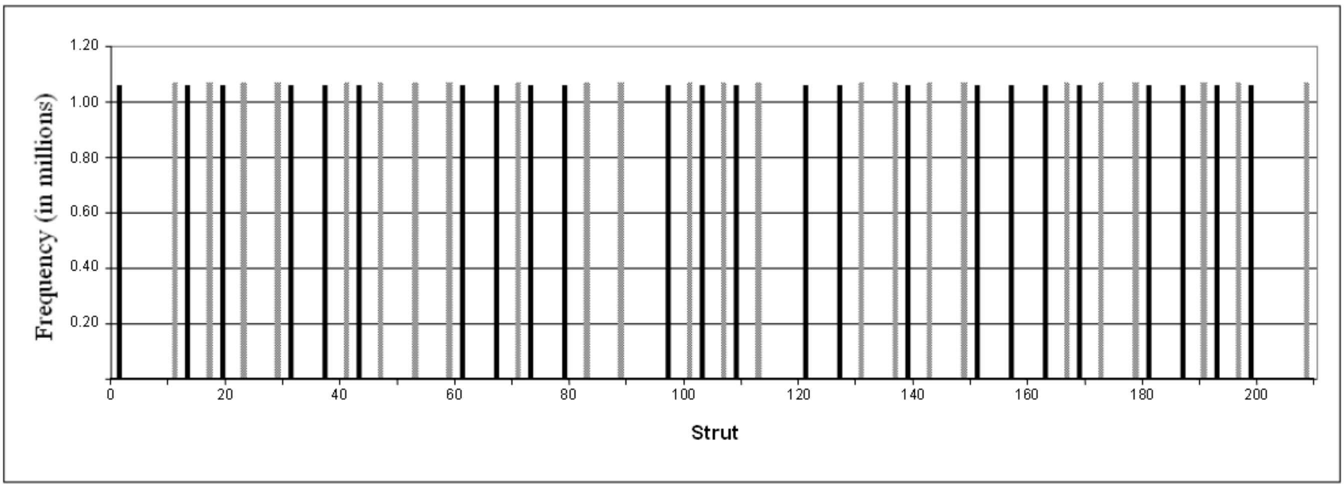

Fig. 8c: P_4#−sieve: distribution of prime numbers < 10^9 based on $6a \pm 1$.

3.3. The Last digit gap conundrum.

In 2016, Standford number theorists, Robert Lemke Oliver and Kannan Soundararajan discovered that the first hundred million consecutive prime numbers, e.g. $\boldsymbol{\pi}(m) = 10^8$, end less frequently with the same digit than other digits. The last digit gap (9, 1) is favored (Fig. 9a).
This find is exceptional because prime numbers have no last digit preference.

	$ld_2 = 1$	$ld_2 = 3$	$ld_2 = 7$	$ld_2 = 9$
$(1, ld_2)$	4,6%	7,4%	7,5%	5,4%
$(3, ld_2)$	6,0%	4,4%	7,0%	7,5%
$(7, ld_2)$	6,4%	6,8%	4,4%	7,4%
$(9, ld_2)$	8,0%	6,4%	6,0%	4,6%

Fig. 9a: Last digit gap occurrence in the first hundred million consecutive prime numbers.

3.3.1. An explanation for the Last digit gap uneven distribution.

An explanation for this uneven distribution of the last digit gap among prime numbers follows from the build of the $P_4\#$−sieve. Fig. 9b shows the last digit gap occurrence in the struts of the $P_3\#$−sieve and the $P_4\#$−sieve. Included are the (9, 1)−combinations of the last strut with the repetition of the first strut, that follows when generating the next primorial sieve.
The $P_3\#$−sieve has no consecutive struts that end with the same last digit. The symmetrical distributed $\varphi(p_3\#) = 8$ struts of the $P_3\#$−sieve are repeated p_4 times. Multiples of p_4 are removed, generating newly last digit combinations (1, 1) and (9, 9) (Fig. 5).

	$P_3\#$−sieve				**$P_4\#$−sieve**			
	$ld_2 = 1$	$ld_2 = 3$	$ld_2 = 7$	$ld_2 = 9$	$ld_2 = 1$	$ld_2 = 3$	$ld_2 = 7$	$ld_2 = 9$
$(1, ld_2)$	0,0%	12,5%	12,5%	0,0%	2,1%	10,4%	12,5%	0,0%
$(3, ld_2)$	0,0%	0,0%	12,5%	12,5%	2,1%	0,0%	10,4%	12,5%
$(7, ld_2)$	12,5%	0,0%	0,0%	12,5%	10,4%	4,2%	0,0%	10,4%
$(9, ld_2)$	12,5%	12,5%	0,0%	0,0%	10,4%	10,4%	2,1%	2,1%

Fig. 9b: Last digit gap occurrence within the struts of the $P_3\#$−sieve and the $P_4\#$−sieve.

Building the struts of the $P_5\#$−sieve using the struts of the $P_4\#$−sieve gives eight times the unique last digit sequence {9, 9, 1, 1} (Fig. 9c). Prime numbers above struts with this sequence have in the beginning a higher chance at a last digit gap combination (9, 1). The influence of this {9, 9, 1, 1} sequence is inherited by every next sieve.

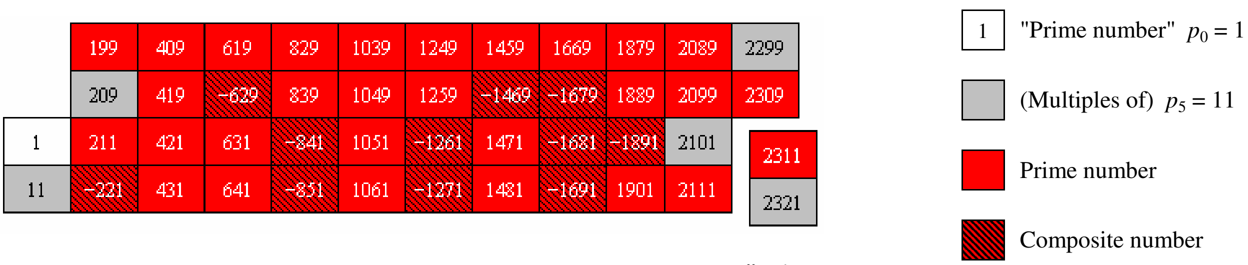

Fig. 9c: The last digit gap sequence {9, 9, 1, 1} in the struts of the $P_5\#$−sieve.

3.3.2. Investigating the difference in the Last digit gap frequency.

In 2016, Robert Lemke Oliver and Kannan Soundararajan were caught off guard by quirks in the final digits of primes.
In the first hundred million consecutive prime numbers, e.g. $\pi(m) = 10^8$ with $m \approx 2 \cdot 10^9$, the number theorists found variations that far exceeded expectations.
See the table which is also displayed in Fig. 9a above.

# of prime numbers				100,000,000
	$ld_2 = 1$	$ld_2 = 3$	$ld_2 = 7$	$ld_2 = 9$
$(1, ld_2)$	4.6%	7.4%	7.5%	5.4%
$(3, ld_2)$	6.0%	4.4%	7.0%	7.5%
$(7, ld_2)$	6.4%	6.8%	4.4%	7.4%
$(9, ld_2)$	8.0%	6.4%	6.0%	4.6%

Investigations into the uniqueness of this find led to Fig. 9d.
The graph shows the frequencies of the Last digit gaps for several values of m, with m the number of consecutive natural numbers. The displayed curves stabilize after the erratic behaviour up to around $m = 10^5$, due to the buildup of the $P_n\#-$sieves.
It is the conjecture that all curves ultimately converge to the expected value of 6,25% (e.g. 1/16) and thus:

Conjecture: "prime numbers have no last digit preference".

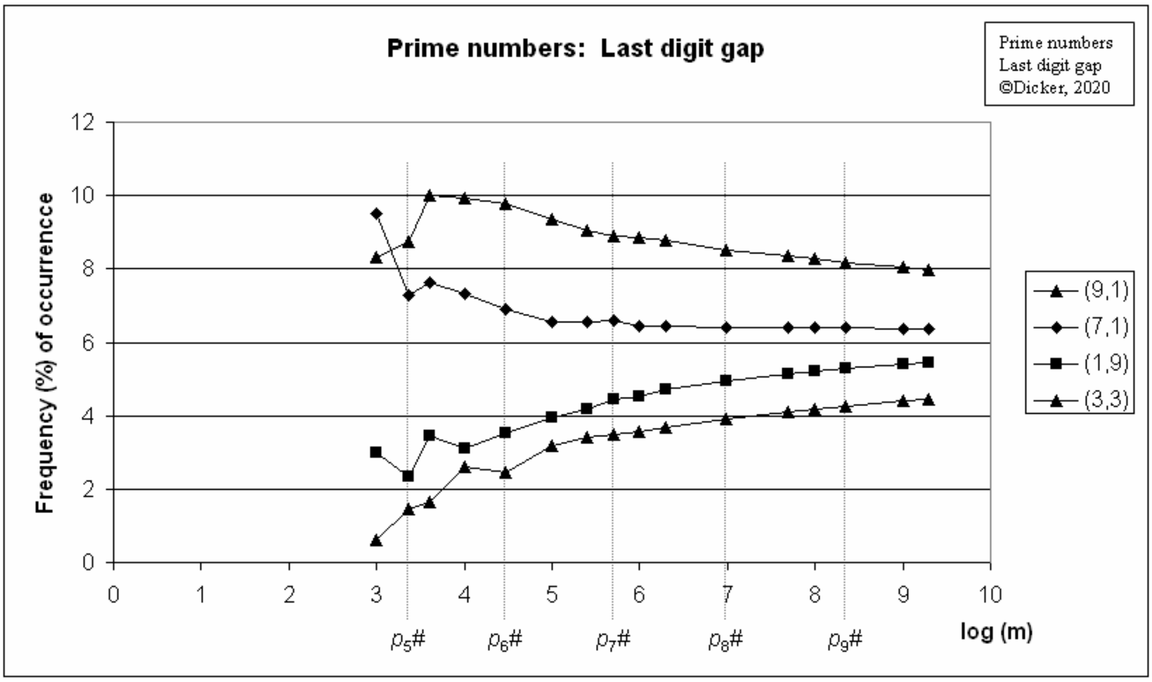

Fig. 9d: Curves with the Last digit gap occurrence up to the first hundred million consecutive prime numbers.

4. Results of the (double) primorial sieve.

The primorial sieve consists of the infinite set of $P_n\#$–sieves. The width of the sieve is equal to the primorial $p_n\#$, the product of the first n prime numbers. All natural numbers sequential arranged on top of the base of the sieve form together a matrix of infinite height.
The $\varphi(p_n\#)$ struts $S(p_n\#)_j$ of the sieve support the columns above which potential prime numbers $g > p_n$ are located, that comply with $g \bmod(p_n\#) \in \{ S(p_n\#)_j \mid 1 \leq j \leq \varphi(p_n\#) \}$. Non-prime numbers with $gcd\,(g,\ p_n\#) \neq 1$ are filtered through the holes in the sieve.

The base of the primorial sieve contains the list of consecutive prime numbers. The discrete upper limit can be extended infinitely far by building ever increasing $P_n\#$–sieves from the previous sieve. The full list $> p_n$ with prime numbers $< p_n\#$ appears when in the last sieve all composite struts are omitted.
This cleanup is done via $S(p_n\#)_j \bullet S(p_n\#)_i < p_n\#$ with $i,\ j > 1$ and $S(p_n\#)_j \leq S(p_n\#)_i$ and not as with the Sieve of Eratosthenes via multiples of all prime numbers $< \sqrt{p_n\#}$.

The double primorial sieve is a method for preliminary filtering of potential prime numbers within all natural numbers. Of the infinite set of natural numbers only $\varphi(p_n\#)\ /\ p_n\#$ numbers are left that could be a prime number. For the final check of a potential prime number $g > p_n$ the division by prime divisors $d \leq \sqrt{g}$ can be approximated by
$d \in \{ S(p_n\#)_j + m \bullet p_n\# \mid 1 \leq j \leq \varphi(p_n\#) \wedge m \in \mathbf{N}_0 \}$.
From the $P_4\#$–sieve onwards the $\varphi(p_n\#)$ struts are almost directly proportional split over the base. There is no need for an indexed array, or binary search.
The double primorial sieves (specifically the $P_4\#$–sieve) offers a platform to further investigate the distribution of prime numbers. Dividing prime numbers into separate groups based on $6a \pm 1$ with $a \in \mathbf{N}^+$ might offer an opening.

The $P_9\#$–sieve with a base of $p_9\# = 223{,}092{,}870$ is the last sieve where 4 Byte integers suffices in the calculations.
By making the $P_9\#$–sieve five times wider all prime numbers $< 10^9$ can be efficiently loaded in memory.
The stretched sieve uses one sequential array with length $5 \bullet p_9 \bullet \varphi(p_8\#) \approx 200 \bullet 10^6$ and 800 MB internal memory.

The effectiveness of the double primorial sieve is the ratio $\varphi(p_n\#)\ /\ p_n\#$, the number of struts (or spokes in the wheel).

n	p_n	$p_n\#$	Number of struts $\varphi(p_n\#)$	Ratio $\varphi(p_n\#)\ /\ p_n\#$	Comment
1	2	2	1	0.5000	odd integers
2	3	6	2	0.3333	integers $6a \pm 1$
3	5	30	8	0.2667	
4	7	210	48	0.2296	
. . .	. . .				
9	23	223,092,870	36,495,360	0.1636	$P_{10}\#$–sieve: ratio is 0.1579

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